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Free-surface breakdown in a rapidly rotating liquid

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It is shown that the wavelets which appear on the inertial wave form of the inner free surface of a fully spun-up cylindrical mass of liquid contained in a vertical, rapidly rotating and gyrating gyrostat are capillary waves. It is further shown that the interaction between these capillary waves and the excited inertial waves is not the mechanism which effects an observed two-period collapse ('breakdown') and reappearance of the free-surface inertial wave form. Rather, the two-period breakdown can be explained by the conjecture that it is a beat phenomenon arising from the interaction of two differently structured inertial wave modes, which have the same frequency at small amplitudes of oscillation of the gyrostat but which, owing to the dependence of the inertial mode frequency on the amplitude of the gyrostatic motion, have slightly different frequencies at larger amplitudes of oscillation of the gyrostat.

1. Introduction

Stewartson (1959) showed that, if there is resonance between the nutational frequency of a liquid-filled top and the frequency of one of the inertial modes of the fully spun-up liquid contained in a cylindrical cavity in the spinning top, then the amplitude of motion of the top may grow. Using an inviscid analysis, he derived an easily calculable expression for the amplitude growth rate of the top. We at the Ballistic Research Laboratories, noting the relevance of the Stewartson analysis to the stability problems of many liquid-filled projectiles, have been continually astonished (and pleased!) at the experimental accuracy of that small amplitude expression for the growth rate. However, as G. N. Ward, in the appendix to the Stewartson paper, pointed out, that expression fails to agree with measurements once the amplitude of motion of the top exceeds two or three degrees.

While verifying that a gyrostat, i.e. a gyroscope with the pivot point located at the centre of mass, containing a cylindrical cavity partially filled with a liquid (Karpov 1965) also may exhibit this anomalous amplitude growth rate when the amplitude of motion exceeds only a few degrees, we stroboscopically observed the free surface (figure 1) of the spinning liquid in the gyrostat with the hope of resolving the anomaly. We noted, with great surprise, that at these 'large' amplitudes two phenomena occurred: (i) many wavelets, i.e. ripples, appeared superimposed on the inertial wave form of the free surface (figure 1); (ii) multi-periodically (depending on the geometry of the cavity and the percentage of fill) the free surface 'broke down', i.e. the wave form continually disappeared (the free surface degenerating into an axisymmetric cylinder) then reappeared, as shown in figure 1. Note an anomaly of the breakdown

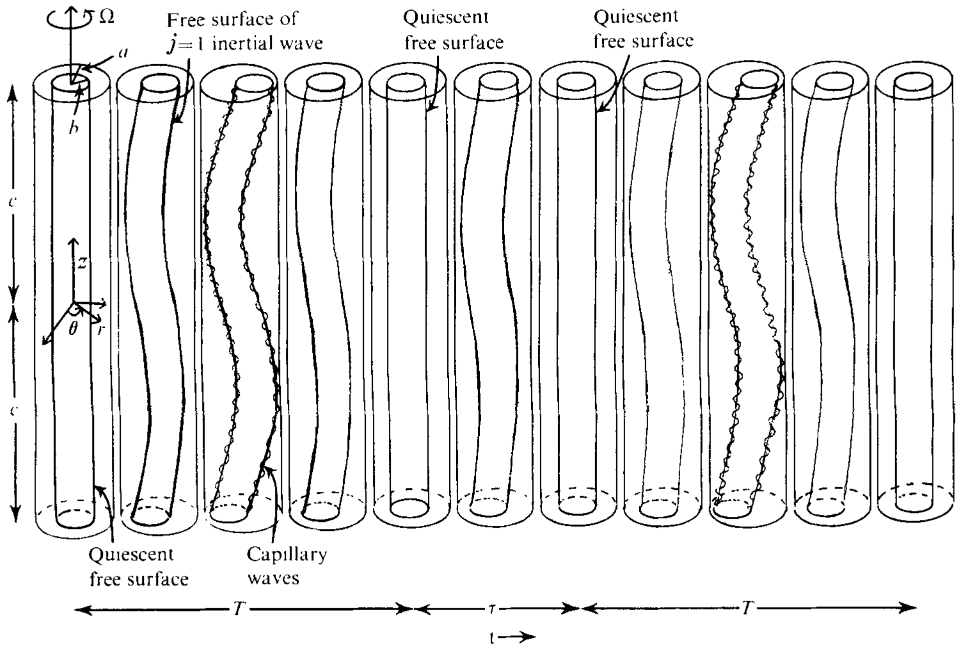

FIGURE 1. Configurations of the free surface of the $j = 1$ inertial wave (starting from the quiescent state and with superposed capillary waves) during successive major (T) and minor (τ) breakdown periods.

process in this figure in that there seem to be two breakdown periods, the larger period T corresponding to breakdown at a rather large amplitude, the shorter period τ corresponding to breakdown at a small amplitude. Scott (1975), having found that 'beating' between two different inertial modes could not account for the breakdown, then unsuccessfully attempted to explain it in terms of interactions between inertial and Rossby waves. No mention was made of the wavelets in that reference for two reasons: (i) it was not known at the time what they were; (ii) whatever they were, they were conjectured to be irrelevant to the breakdown phenomenon.

In view of the well-known interaction between capillary and gravity waves (Lamb 1932, pp. 456–470), we are now somewhat embarrassed over the failure of the Scott reference to mention those wavelets, to conjecture that they might be capillary waves and, further, to conjecture that the free-surface breakdown might be due to an interaction between these capillary waves and the ever-present inertial waves. As an afterthought, this last conjecture seemed not unreasonable, for if the presence of capillary waves on the free surface of the rotating liquid could alter the inertial mode frequency without altering the gyrostatic frequency, a state of resonance would no longer exist between one of the inertial modes and the gyrostatic motion. Hence the altered inertial mode would die with the concomitant disappearance of the inertial wave form. Then, since the free surface would be a concentric cylinder, surface tension could no longer induce capillary waves, and the 'pure' inertial mode could then be re-excited. However, as we shall show in the analysis, calculations of the influence of surface tension on the dominant inertial mode frequency gave an effect an order of magnitude too small. It then occurred to us that the periodic breakdown might still be a beat phenomenon (in spite of our previous dismissal of any effect of beating between two different inertial modes) since the particular cavity in which we observed the breakdown was such that two of the inertial modes, though having different structure, had the same frequency. It then became natural to inquire whether surface tension could 'split' the modes, i.e. could alter the frequency of the mode with the more complicated structure more significantly than the mode with less complicated structure. If so, then one would have two modes of nearly equal frequency in the cavity, the resulting beat frequency hopefully coinciding with the collapse frequency of the free surface. Again calculations showed that surface tension was not strong enough to effect the requisite frequency split.

Only after the failure of this last surmise did the thought occur to us that the amplitude dependence of the inertial modes (Scott 1975) might affect the two frequencies differently, i.e. 'split' the modes, thus giving rise to the beat phenomenon. Calculations presented here seem to indicate that this is indeed the mechanism for the multi-period free-surface collapse.

Since there seems to be no prior publication of observations of capillary waves on a rotating liquid, since the interaction of inertial and capillary waves seems to have attracted no attention and since there seems to be no prior publication of the effects of an interaction between two different inertial modes, we present the observations and analysis here.

2. The interaction between inertial and capillary waves *vis-à-vis* the free-surface breakdown

In order to determine the inertial wave frequencies of the liquid in the rotor (of the gyrostat) shown in figure 1, we follow Stewartson (1959) and all of his assumptions and write

$$\partial \mathbf{q}/\partial t + 2\mathbf{\Omega} \times \mathbf{q} = -\nabla P/\rho, \tag{1}$$

where $\mathbf{q}$ is velocity, $\mathbf{\Omega}$ is the angular velocity, P is the reduced pressure and ρ is the density. Setting $\mathbf{q} = \mathbf{Q}\, e^{st}$ and $P = p\, e^{st}$ in (1) and then solving for $\mathbf{Q}$, we have

$$\mathbf{Q} = \{2\mathbf{\Omega} \times \nabla p/s\rho - \nabla p/\rho - 4\mathbf{\Omega\Omega}\,.\,\nabla p/s^2\rho\}/s(1+4\Omega^2/s^2). \tag{2}$$

Using this expression in the continuity equation $\nabla\,.\,\mathbf{Q} = 0$, we have

$$\nabla^2 p + (2\Omega/s)^2\, \partial^2 p/\partial z^2 = 0, \tag{3}$$

a partial differential equation involving the pressure p which is to be solved subject to appropriate boundary conditions.

To determine the frequencies of the inertial waves, we can let the rotor have zero transverse velocity. Then the boundary condition is $\mathbf{Q}\,.\,\mathbf{n} = 0$, which in terms of the pressure p becomes, at the end faces of the cylinder,

$$\{\partial p/\partial z\}_{z=\pm c} = 0. \tag{4}$$

At the cylinder walls, the boundary condition $\mathbf{Q}\,.\,\mathbf{n} = 0$ becomes, in terms of the pressure p,

$$\{\partial p/\partial r + (2\Omega/sr)\, \partial p/\partial \theta\}_{r=a} = 0. \tag{5}$$

At the free surface (figure 1), the equation for which we shall take as

$$r = b\{1 + \eta(\theta, z)\, e^{st}\}, \tag{6}$$

the kinematic boundary condition $dF/dt = 0$, where $F \equiv r - b\{1 + \eta(\theta, z)\, e^{st}\}$, yields, at first order in η,

$$s\eta = -\{\partial p/\partial r + (2\Omega/sr)\, \partial p/\partial\theta\}_{r=b}/sb\{1 + (2\Omega/s)^2\}. \quad (7)$$

The dynamical boundary condition at the free surface yields, at first order in η,

$$pe^{st} = -\rho\Omega^2 b^2 \eta\, e^{st} + \sigma(1/b + b\, e^{st}\, \partial^2\eta/\partial z^2 + e^{st}\, \eta/b + e^{st}\, \partial^2\eta/b\, \partial\theta^2), \quad (8)$$

where σ is the surface tension, where the second term in the parentheses is the free-surface curvature in the z direction and where the other terms in the parentheses give the free-surface curvature in the θ direction. Differentiating (8) with respect to time and using (7), we have, finally, as the appropriate expression for the boundary condition at the free surface

$$\{s^2 + 4\Omega^2 - [\Omega^2 b - \sigma(\partial^2/\rho\, \partial z + 1/\rho b + \partial^2/b^2\rho\, \partial\theta^2)]\, (\partial/\partial r + 2\Omega\, \partial/sr\, \partial\theta)\}p = 0 \quad \text{at} \quad r = b. \quad (9)$$

A solution of (3) that satisfies (4), that has the requisite θ dependence to allow the gyrostatic motion to effect the inertial oscillations and that has sufficient undetermined constants for the satisfaction of (5) and (9) is

$$p = \sum_j e^{i\theta}\{AJ_1(\alpha r) + BY_1(\alpha r)\} \sin k_j z, \quad (10)$$

where A and B are constants, $J_1(\alpha r)$ and $Y_1(\alpha r)$ are Bessel functions and Neumann functions of the first order, $\alpha = (2j+1)\, (\pi/2c)\, [-(1 + 4\Omega^2/s^2)]^{\frac{1}{2}}$ and $k_j = (2j+1)\pi/2c$.

Substituting (10) into (5) and (9) respectively gives

$$(d/dr + 2\Omega i/sr)\{AJ_1(\alpha r) + BY_1(\alpha r)\}_{r=a} = 0 \quad (11)$$

and $$\{-(\Omega^2 b + \sigma k_j^2/\rho)\, (d/dr + 2\Omega i/sr) + s^2 + 4\Omega^2\}\{AJ_1(\alpha r) + BY_1(\alpha r)\}_{r=b} = 0. \quad (12)$$

Dividing (11) and (12) by $\Omega^2 b$, using recursion relations involving the derivatives of the Bessel and Neumann functions and invoking the condition that A and B be non-zero, we get the frequency equation

$$(J_0, J_1)_a \{Y_0, Y_1\}_b - (Y_0, Y_1)_a \{J_0, J_1\}_b \equiv G(s) = 0, \quad (13)$$

where $$(J_0, J_1)_a \equiv \alpha J_0(\alpha a) + (2\Omega i/s - 1)\, J_1(\alpha a)/a$$

and $$\{Y_0, Y_1\}_b \equiv \{(1 - 2\Omega i/s)\, (1 + \delta) + 4 + s^2/\Omega^2\}\, Y_1(\alpha b) - \alpha b(1 + \delta)\, Y_0(\alpha b).$$

Similar expressions exist for $(Y_0, Y_1)_a$ and $\{J_0, J_1\}_b$, where $\delta = (\sigma/b\rho\Omega^2)\, k_j^2$.

Not only does (13) yield values of s that are, of course, purely imaginary, but also, if we set $\delta = 0$ in (13), i.e. set $\sigma = 0$, and let the corresponding value of s be s_0, there results the frequency equation (as a function of the parameters $c/a(2j+1)$ and b^2/a^2) from which Stewartson (1959) determined the inertial mode frequencies. The value of $2j+1$ is the number of half sine waves that can be fitted between the ends of the cavity and $1 - b^2/a^2$ is the fill ratio.

The various values of the s's, i.e. the frequencies, can be calculated from the fact that, in (13), $\delta \ll 1$ for our experiments; hence s_0 is approximately a solution of the equation and we can write, for example,

$$s = s_0 + \Delta s_0, \quad (14)$$

where Δs_0, the frequency change due to the effect of surface tension, is given by Newton's root approximation method as

$$\Delta s_0 = -G(s_0)\,(dG(s_0)/ds_0)^{-1}, \tag{15}$$

where
$$dG(s_0)/ds_0 = \{Y_0, Y_1\}_{b,s_0} d(J_0, J_1)_{a,s_0}/ds_0 + (J_0, J_1)_{a,s_0} d\{Y_0, Y_1\}_{b,s_0}/ds_0 - (Y_0, Y_1)_{a,s_0} d\{J_0, J_1\}_{b,s_0}/ds_0 - \{J_0, J_1\}_{b,s_0} d(Y_0, Y_1)_{a,s_0}/ds_0. \tag{16}$$

Let us now consider a cylindrical cavity that has a height-to-diameter ratio, i.e. a value of c/a, of 3·08. Then from Stewartson's tables it can be shown that the liquid in such a cavity can support two differently structured inertial modes (the $j = 1$ and the $j = 3$ modes, corresponding to three and seven half sine waves) with the same frequency, 0·947Ω. Using (15), we find that the effect of surface tension on the $j = 1$ mode at this $c/a = 3{\cdot}08$ for a cavity containing water or low viscosity oil is different from the effect on the $j = 3$ mode, but the difference is an order of magnitude too small to produce the observed beats. Furthermore, increasing the angular speed of the gyrostat theoretically increases the beat period owing to the interaction of capillary and inertial waves whereas experimentally the beat period is reduced.

Since the theoretical prediction of the functional dependence of the breakdown period on the angular speed of the gyrostat is in the wrong direction (in addition to being numerically far removed from the experimental value) and since observations showed that the breakdown period for the oil was experimentally indistinguishable from that for water (the surface tension of the water was 74 dynes/cm, that of the oil 23 dynes/cm), it would seem that surface tension is not relevant in the breakdown process. As an afterthought, the negligible effect of surface tension does not seem physically unreasonable; for inertial waves are internal waves, whereas capillary waves are surface waves.

As conclusive evidence that the wavelets are, indeed, capillary waves and not significantly related to the breakdown process, we mention that the addition of dishwasher soap (a surfactant that reduces surface tension without any comparable effect on the viscosity) markedly reduced the size of the wavelets and delayed their appearance without noticeably affecting the breakdown period.

3. The interaction of the amplitude-modulated inertial waves *vis-à-vis* the free-surface breakdown

If we are correct that the free-surface breakdown, though not surface-tension dominated, is still a beat phenomenon, then the liquid in cavities having height-to-diameter ratios c/a of 3·045 and 3·711 should not exhibit free-surface breakdown. This follows from the fact, deducible from Stewartson's tables, that two differently structured inertial modes having equal frequencies cannot coexist in such cavities. Hence the mechanism for beating does not exist and if our conjecture is correct, there should be no free-surface breakdown even though capillary waves should appear. We confirmed this experimentally: capillary waves appeared on the free surface for each cavity, but no breakdown occurred until the amplitude reached 7° or 8° (compared with 2° or 3° for the $c/a = 3{\cdot}08$ cavity). Furthermore, once the breakdown had occurred, the free-surface wave form did not reappear: rather, the collapsed surface, though basically cylindrical, remained in a state of agitation, vastly different from the perfectly quiescent surface that resulted from breakdown in the $c/a = 3{\cdot}08$ cavity. This would seem to indicate that at such amplitudes the inertial wave frequency has changed sufficiently that the system is no longer a resonant one (Scott 1975).

The above experimental confirmation that the inertial modes are amplitude dependent (owing to the failure of the angular velocity vector of the liquid to remain coincident with the angular velocity vector of the rotor during large amplitude motion of the gyrostat, thus causing the liquid to 'see' a modified cylindrical cavity) finally led us to inquire whether or not such an amplitude effect on the frequency was different for the two equal-frequency modes of the $c/a = 3{\cdot}08$ cavity. If so, we reasoned, such a differential effect could result in the two modes having their frequencies sufficiently split at the large amplitudes to effect the beat phenomenon. Using equation (58) of Scott (1975) to determine the frequency shift, we find that the $j = 1$ mode indeed 'sees' a different cavity at large amplitudes from the one which the $j = 3$ mode 'sees', the resulting frequency shift being from $0{\cdot}947\Omega$ to $0{\cdot}9394\Omega$ for the $j = 1$ mode and from $0{\cdot}947\Omega$ to $0{\cdot}9437\Omega$ for the $j = 3$ mode. Representing the pressure waves associated with these frequencies as the sum of two sine terms [see (10)] and letting the frequency shift in the $j = 1$ mode be Δs_1 and the frequency shift in the $j = 3$ mode be Δs_3, we have at large amplitudes

$$p \sim A_{j=1} \sin\{\theta + (s + \Delta s_1)t\} + A_{j=3} \sin\{\theta + (s + \Delta s_3)t\}. \tag{17}$$

Since the relationship between the amplitudes is $A_{j=1} = \frac{49}{9} A_{j=3}$ (see (3.3) of Stewartson 1959), we have, approximately,

$$p \sim 2A_{j=1}\{\cos \tfrac{1}{2}(\Delta s_1 - \Delta s_3)t - 0{\cdot}408\} \sin\{\theta + (s + \Delta s_3)t\}. \tag{18}$$

In (18), the coefficient of the sinusoidal term vanishes at t_0, t_1, t_2, t_3, t_4, etc., where

$$\tfrac{1}{2}(\Delta s_1 - \Delta s_3)t_0 = \cos^{-1} 0{\cdot}408, \quad \tfrac{1}{2}(\Delta s_1 - \Delta s_3)t_1 = 2\pi - \cos^{-1} 0{\cdot}408,$$

$$\tfrac{1}{2}(\Delta s_1 - \Delta s_3)t_2 = 2\pi + \cos^{-1} 0{\cdot}408, \quad \tfrac{1}{2}(\Delta s_1 - \Delta s_3)t_3 = 4\pi - \cos^{-1} 0{\cdot}408,$$

$$\tfrac{1}{2}(\Delta s_1 - \Delta s_3)t_4 = 4\pi + \cos^{-1} 0{\cdot}408, \quad \text{etc.}$$

Hence T, the major period, is given by

$$T = (4\pi - 4\cos^{-1} 0{\cdot}408)/(\Delta s_1 - \Delta s_3) \tag{19}$$

and τ, the minor period, is given by

$$\tau = 4\cos^{-1} 0{\cdot}408/(\Delta s_1 - \Delta s_3). \tag{20}$$

Expressions (19) and (20) yield, for the $c/a = 3{\cdot}08$ cavity,

$$T = 5{\cdot}9\,\text{s}, \quad \tau = 3{\cdot}4\,\text{s} \quad \text{for} \quad \Omega = 300\,\text{r.p.m.},$$

$$T = 4{\cdot}43\,\text{s}, \quad \tau = 2{\cdot}55\,\text{s} \quad \text{for} \quad \Omega = 4000\,\text{r.p.m.}$$

The experimental values of these periods were approximately 8, 2, 5 and 1 s, respectively. This reasonable agreement between theory and experiment lends credibility to our conjecture concerning the presence and source of the beat phenomena.

Cavities with two different modes of equal frequency are not common. Diligently searching the Stewartson tables, we find that for a $c/a = 2{\cdot}583$ cavity 60 % filled the $j = 1$ and $j = 4$ modes both have frequencies of $0{\cdot}98\Omega$. The large amplitude effect on these shifts the $j = 1$ mode from $0{\cdot}98\Omega$ to $0{\cdot}97\Omega$ and the $j = 4$ mode from $0{\cdot}98\Omega$ to $0{\cdot}975\Omega$, giving major and minor breakdown periods of 5·2 and 2·83 s at 3000 r.p.m. Again, these values are reasonably close to the approximate experimental values of 7 and 2 s, respectively.

4. Summary

The wavelets which appear on the inner free-surface inertial wave in a rapidly rotating liquid partially filling an oscillating gyrostat are capillary waves, but, unlike the situation for capillary waves on the free surface of a liquid supporting gravity waves, there is negligible dynamical interaction between the inertial and capillary waves.

The sensitivity of the inertial wave frequency to the geometry of the cavity is manifested by the fact that at 'large' amplitudes of motion of the gyrostat the liquid no longer 'sees' a cylindrical cavity. This differentially shifts the frequencies, so that, if at small amplitudes of motion of the gyrostat two differently structured inertial wave modes have equal frequencies, they will not have equal frequencies at larger amplitudes. The combination of these two waves of slightly different frequencies can give rise to the phenomenon of beats, one visual consequence of which can be a two-period collapse of the wave form of the free surface.

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